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A Note on Noncommutative D-Brane Actions

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Abstract

We use the nonabelian action of N coincident D(-1) branes in constant background fields, in the $N \to \infty$ limit, to construct noncommutative D-brane actions in an arbitrary noncommutative description and comment on tachyon condensation from this perspective.

1 Introduction and Review

It is a well known result that from the non-abelian Born-Infeld action of infinitely many D(-1) instantons one can construct the background independent, $\Phi = -B$, description of noncommutative D-branes. Similarly, in [?, ?, ?], the same type of equivalence was shown for the Chern-Simons terms. In [?], it has been remarked that by placing D(-1) instantons in a constant B-field one can construct noncommutative D-branes with arbitrary noncommutativity. In this note, we clarify this point by starting from the action of N coincident D(-1) instantons in a constant B-field as given by [?, ?]. We show that such actions lead us to construct D-brane actions in an arbitrary noncommutative description. The map relating the Born-Infeld terms is seen to be consistent with the map relating the Chern-Simons terms. We use this result to study tachyon condensation on a noncommutative D-brane with arbitrary θ.

We will now review some relevant results of [?] and [?]. For concreteness, we will assume Euclidean space-time and maximal rank constant B-field along the directions of a Dp-brane. We use the convention where $2\pi\alpha' = 1$. Then the world-volume Dp-brane action can be described in noncommutative variables, i.e. $[x^i, x^j] = i\theta^{ij}$, as

$$\hat{S}_{BI} = \frac{(2\pi)^{\frac{1-p}{2}}}{G_s} \int d^{p+1}x \sqrt{det(G + \hat{F} + \Phi)}, \tag{1}$$

where the * product is implicit in the above equation. For abelian and constant F, the Seiberg-Witten transformations relating F to $\hat{F}$ are given by

$$F = \hat{F}\frac{1}{1-\theta\hat{F}}, \qquad \hat{F} = \frac{1}{1+F\theta}F. \tag{2}$$

For every closed string background characterized by the NS-NS 2-form B, the closed string metric g, and the closed string coupling constant g_s, there is a continuum of descriptions given by a choice of Φ. The open string metric G, the open string coupling constant G_s and the noncommutativity parameter θ can be expressed in terms of closed string variables as follows:

$$\frac{1}{G+\Phi}+\theta=\frac{1}{g+B}, \tag{3}$$

$$G_s=g_s\left(\frac{det(G+\Phi)}{det(g+B)}\right)^{\frac{1}{2}}.$$

Finally, let us review the main results of [?]. The non-abelian Born-Infeld action describing N (Euclidean) coincident Dp-branes in a closed string background defined by ϕ, B' and g is

$$S_{BI}=\frac{(2\pi)^{\frac{1-p}{2}}}{g_s}\int d^{p+1}\sigma Str\left(e^{-\phi}\sqrt{det(P[E_{ab}+E_{ai}(M^{-1}-\delta)^{ij}E_{jb}]+F_{ab})det(M^i_j)}\right), \tag{4}$$

where $E\equiv g+B'$ and ϕ is the bulk dilaton. Furthermore, i,j are indices for the transverse coordinates, a,b are indices for the coordinates parallel to the D-brane, and the X's are $N\times N$ matrices representing the transverse displacements expressed in the static gauge. We also defined[1]

$$M^i_j\equiv\delta^i_j-i[X^i,X^k]E_{kj}. \tag{5}$$

For the non-abelian Chern-Simons action, we have

$$S_{CS}=\mu_p\int Str\left(P[e^{-i(i_Xi_X)}(\sum C^{(n)}e^{B'})]e^F\right), \tag{6}$$

where μ_p is the RR charge of a Dp-brane. In the aforementioned actions, the bulk fields should be considered functionals of the $N\times N$ matrices X, and the trace should be symmetrized between all expressions of the form F_{ab}, D_aX^i, $[X^i,X^j]$, and X^k. However, since we are only going to consider D(-1) instantons in constant background fields, these details are irrelevant for our purposes.

More precisely, in the next two sections we consider an infinite number of D(-1) instantons with $\phi=0$ and where g and B' are constants. The presence of the B' field will allow us

[1] Unlike in [?], we used the convention $F_{ab}=\partial_aA_b-\partial_bA_a-i[A_a,A_b]$ in order to be consistent with the definition of $\hat{F}$ in [?].

to construct D-brane actions in an arbitrary noncommutative description. In section 2, we show that the Born-Infeld action of D(-1) instantons in a constant B' field naturally leads to NC Born-Infeld action, where the B field is identified as $B = B' + \theta^{-1}$ for arbitrary noncommutativity parameter θ. Having shown this, the nonabelian generalization of the Chern-Simons action for an infinite number of D(-1) instantons should correspond to the NC Chern-Simons action in the same noncommutative description as the BI action. This fact is confirmed in section 3. In section 4, we comment on tachyon condensation using the connection to the matrix model. We find that in the presence of a constant B-field, the vacuum becomes non-commutative.

2 Born-Infeld Action

In this section, we follow the line of thought in [?] and derive the equivalence of the nonabelian BI action of an infinite number of D(-1) instantons and the BI action of a noncommutative Dp-brane in a general noncommutative description. First consider the nonabelian BI action of N D(-1) branes ($N \to \infty$) in a constant B'-field:

$$S_{BI} = \frac{2\pi}{g_s} Str \sqrt{det_{ij} \left(\delta_i{}^j - i(g + B')_{ik}[X^k, X^j] \right)}. \tag{7}$$

We are interested in a particular classical configuration given by

$$[x^i, x^j] = i\theta'^{ij}. \tag{8}$$

The degrees of freedom on the noncommutative Dp-brane arise by expanding the matrix variable X^i around this classical configuration as follows:

$$X^i = x^i + \theta'^{ij} \hat{A}'_j. \tag{9}$$

Then, we have

$$i[X^i, X^j] = (\theta' \hat{F}' \theta' - \theta')^{ij}, \tag{10}$$

where

$$\hat{F}'_{ij} = -i\theta'^{-1}_{ik}[x^k, \hat{A}'_j] + i\theta'^{-1}_{jk}[x^k, \hat{A}'_i] - i[\hat{A}'_i, \hat{A}'_j]. \tag{11}$$

We can reexpress Tr over the Hilbert space as an integral over the volume of noncommutative space by replacing

$$Tr \to \frac{1}{(2\pi)^{\frac{(p+1)}{2}} Pf\theta'} \int d^{p+1}x, \tag{12}$$

where $Pf\theta'$ is the Pfaffian. We write the action in terms of new variables,

$$\begin{aligned} S_{BI} &= \frac{(2\pi)^{\frac{1-p}{2}}}{g_s} \int \frac{d^{p+1}x}{Pf\theta'} \sqrt{det\left[1 - (g+B')(\theta'\hat{F}'\theta' - \theta')\right]} && (13) \\ &= \frac{(2\pi)^{\frac{1-p}{2}}}{g_s} \int d^{p+1}x \sqrt{det\left[\theta'^{-1} - (g+B')(\theta'\hat{F}' - \mathbf{1})\right]} && (14) \\ &= \frac{(2\pi)^{\frac{1-p}{2}}}{g_s} \int d^{p+1}x \sqrt{det\left[g + B' + \theta'^{-1} - (g+B')\theta'\hat{F}'\right]}. && (15) \end{aligned}$$

We would like to compare this with the BI action of a noncommutative Dp-brane in a description with the same noncommutativity parameter θ which appears in the above action. The NC BI action for a Dp-brane is

$$S_{NCBI} = \frac{(2\pi)^{\frac{1-p}{2}}}{G_s} \int d^{p+1}x \sqrt{det\left(G + \hat{F} + \Phi\right)}. \tag{16}$$

Reexpressing it in terms of closed string variables by using the relations (3) gives us

$$\begin{aligned} S_{NCBI} &= \frac{(2\pi)^{\frac{1-p}{2}}}{g_s} \frac{\sqrt{det(g+B)}}{\sqrt{det(G+\Phi)}} \int d^{p+1}x \sqrt{det\left(G + \Phi + \hat{F}\right)} && (17) \\ &= \frac{(2\pi)^{\frac{1-p}{2}}}{g_s} \int d^{p+1}x \sqrt{det\left(g + B + (g+B)\frac{1}{G+\Phi}\hat{F}\right)} && (18) \\ &= \frac{(2\pi)^{\frac{1-p}{2}}}{g_s} \int d^{p+1}x \sqrt{det\left(g + B + (1 - (g+B)\theta)\hat{F}\right)}. && (19) \end{aligned}$$

We observe that (7) agrees with (16) once we make the following identifications:

$$\theta = \theta', \quad \hat{F} = \hat{F}', \quad B = B' + \theta'^{-1}. \tag{20}$$

Notice that here θ is a free parameter, not fixed to be B^{-1} as in [?]. By identifying B' in the nonabelian action for N D(-1) instantons ($N \to \infty$) with $B - \theta^{-1}$, we can go to the noncommutative description of Dp-brane with arbitrary noncommutativity parameter θ. It is interesting to note that Φ takes the following form in matrix-model-like variables:

$$\Phi = -\theta^{-1}\left(1 + (g + B')^{-1}_{A}\theta^{-1}\right), \tag{21}$$

where A denotes antisymmetrization.

3 Chern-Simons Action

If the nonabelian BI action for an infinite number of D(-1) instantons in a constant B' field gives rise to the NC BI action with $B = B' + \theta^{-1}$ and noncommutativity parameter θ, then we should expect the same identification relates the Chern-Simons term of the nonabelian action with that of the NC theory. This is precisely what occurs, and the Chern-Simons action for a Dp-brane with a constant B field and noncommutativity θ can be expressed as the nonabelian CS action for an infinite number of D(-1) branes in a constant B' field given by [?]

$$S_{CS} = \frac{2\pi}{g_s} Str\left[e^{-i(\mathrm{i}_X \mathrm{i}_X)}\sum_n C^{(n)} e^{B'}\right], \quad B' = B - \theta^{-1}. \tag{22}$$

Here i_X acts on an n-form $\omega^{(n)}$ as

$$\mathrm{i}_X \omega^{(n)} = \frac{1}{(n-1)!} X^{\nu_1} \omega^{(n)}_{\nu_1 \nu_2 \ldots \nu_n} dx^{\nu_2} \ldots dx^{\nu_n}. \tag{23}$$

This provides a natural explanation of the rather surprising result recently derived by [?], where they express an arbitrary NC CS action in terms of matrix-model like variables, which turns out to be identical to (22). For simplicity, we follow the proof of [?] to show that the nonabelian action gives rise to the NC action for D9-branes, where we can ignore transverse scalar fields. In that case, the NC CS action is given by [?, ?]

$$S_{NCCS} = \mu_9 \int_x \sqrt{det(1 - \theta F)} \sum_n C^{(n)} e^{B + \hat{F}(1 - \theta\hat{F})^{-1}}, \tag{24}$$

where $\mu_9 = (2\pi)^{-4}/g_s$ is the RR charge of a BPS D9-brane. In terms of $Q = -\theta + \theta\hat{F}\theta = -\frac{1}{F+\theta^{-1}}$, (24) can be expressed as

$$S_{NCCS} = \mu_9 \int_x \sqrt{det(1-\theta F)} \sum_n C^{(n)} e^{B'} e^{-Q^{-1}}. \tag{25}$$

The nonabelian CS action for an infinite number of D(-1) instantons (22) naturally leads to the NC CS action for Dp-branes (24). Expanding the action (22) and using the fact that $i[X,X] = Q$ give terms of the form

$$\frac{2\pi}{g_s} Tr \left[\frac{(10-2r)!}{2^{5-r}(s-r)!(5-r)!2^{s-r}(10-2s)!} Q^{i_{2r+1}i_{2r+2}} ... Q^{i_9 i_{10}} B'_{[i_{2r+1}i_{2r+2}} ... B'_{i_{2s-1}i_{2s}} C^{(10-2s)}_{i_{2s+1}...i_{10}]} \right], \tag{26}$$

where [...] denotes antisymmetrization and $5 \geq s > r \geq 0$. Employing the identity (12), one gets

$$\mu_9 \int d^{10}x \frac{(10-2r)!}{2^{5-r}(5-r)!(s-r)!2^{s-r}(10-2s)! Pf\theta} Q^{i_{2r+1}i_{2r+2}} ... Q^{i_9 i_{10}} \times \tag{27}$$

$$B'_{[i_{2r+1}i_{2r+2}} ... B'_{i_{2s-1}i_{2s}} C^{(10-2s)}_{i_{2s+1}...i_{10}]}.$$

Finally, the above expression can be simplified to

$$\mu_9 \int d^{10}x \frac{PfQ(-1)^r}{Pf\theta 2^s r!(s-r)!(10-2s)!} \epsilon^{i_1...i_{10}} Q^{-1}_{i_1 i_2} ... Q^{-1}_{i_{2r-1}i_{2r}} B'_{[i_{2r+1}i_{2r+2}} ... B'_{i_{2s-1}i_{2s}} C^{(10-2s)}_{i_{2s+1}...i_{10}]}. \tag{28}$$

One can immediately see that (28) are the terms coming from the expansion of (24). We have shown that our claim holds for the special case $p = 9$. The general case has been already considered in [**?**].

Up to now, we have restricted the Ramond-Ramond fields to be constants, but we can generalize our procedure to the case where the Ramond-Ramond fields are varying by writing the fields as fourier transforms[2] such that

$$S_{CS} = \frac{2\pi}{g_s} \int d^{10}q Str \left[e^{-i(\mathrm{i}_X \mathrm{i}_X)} \sum_n C^{(n)}(q) e^{B'} e^{iq\cdot X} \right], \quad B' = B - \theta^{-1}. \tag{29}$$

[2]See [**?**] for how to relate the currents expressed in matrix model language to those in noncommutative gauge theory.

To conclude, motivated by the identification relating the nonabelian BI action of D(-1) instantons to the BI action of Dp-branes in the last section, we have proposed and verified that the NC CS action of a Dp-brane with arbitrary noncommutativity and varying Ramond-Ramond fields can be derived from considering the nonabelian CS action for an infinite number of D(-1) branes after identifying $B' = B - \theta^{-1}$.

4 Tachyon Condensation

In this section, we study tachyon condensation in open string theory via the matrix model connection as in [?], [?]. Using the results of the previous sections, we study this from an arbitrary noncommutative description. The effective action on a single noncommutative D-brane is

$$\frac{(2\pi)^{\frac{1-p}{2}}}{G_s}\int d^{p+1}x\left[V(T)\sqrt{det\left(G+\hat{F}+\Phi\right)}+\sqrt{G}f(T)G^{ij}D_iTD_jT+...\right]. \tag{30}$$

In terms of matrix-model-like variables, Φ is given by (21), while the open string metric is $G = -\theta^{-1}(g+B')^{-1}_S\theta^{-1}$ where S denotes symmetrization. Thus the effective action can be written as

$$\frac{2\pi}{g_s}Tr\left[V(T)\sqrt{det_{ij}\left(\delta_i^{\ j}-i(g+B')_{ik}[X^k,X^j]\right)}-\right. \tag{31}$$

$$\left.f(T)\sqrt{\frac{g}{g-B'}}(g-B'g^{-1}B')_{ij}[X^i,T][X^j,T]\right].$$

Following [?], we assume that $V(T)$ has a unique minimum at $T = T_c$ (T_c proportional to the unit matrix). The end-point of tachyon condensation obtained by minimizing the Born-Infeld term[3] is characterized by $X = X_c$ satisfying

$$[X_c, X_c] = -i(g+B')^{-1}_A. \tag{32}$$

We expect this minimum to be exact, in the sense that even if there are corrections to the symmetrized-trace proposal for non-abelian Born-Infeld action, these corrections are of the

[3]One can write $det\left(1-i(g+B')[X,X]\right) = det(g+B')det\left((g+B')^{-1}_S+(g+B')^{-1}_A-i[X,X]\right)$.

type $[F, F]$ and DF, so they are irrelevant for our solution. For $B' = 0$, (32) implies that the vacuum is commutative. In the presence of a non-zero constant NS-NS field it changes to a non-commutative state given by (32).

5 Discussion

It is interesting to analyze the Seiberg-Witten limit in this context. In this limit, using the conventions of [**?**], we have

$$\begin{aligned} g &\sim \epsilon, \\ \alpha' &\sim \epsilon^{\frac{1}{2}}, \\ B &= B^{(0)} + \epsilon B^{(1)} + O(\epsilon^2), \\ \theta^{-1} &= B^{(0)} + O(\epsilon). \end{aligned} \tag{33}$$

Keeping track of α', we see that $2\pi\alpha' B' = 2\pi\alpha'(B - \theta^{-1})$ scales as $O(\epsilon^{3/2})$ and $\frac{[X,X]}{2\pi\alpha'}$ goes like $O(\epsilon^{-1/2})$. Hence, in the Seiberg-Witten limit, the Born-Infeld action reduces to

$$S_{BI} = \frac{\pi}{g_s}\left[iB'_{ij}Tr[X^i, X^j] - \frac{1}{2(2\pi\alpha')^2}g_{ik}g_{jl}Tr\left([X^i, X^j][X^k, X^l]\right)\right] + O(\epsilon^{1/2}) + constant, \tag{34}$$

where the second term is the usual potential of the matrix model. Furthermore, the non-abelian Chern-Simons action takes the standard matrix model form

$$S_{CS} = \frac{2\pi}{g_s} Str\left[e^{-\frac{i}{2\pi\alpha'}(\mathrm{i}_X \mathrm{i}_X)} \sum_n C^{(n)}\right], \tag{35}$$

where we assumed appropriate scaling of RR potentials, $C^{(n)}$, such that the limit is well-defined.

Finally, let's remark that since $B = B' + \theta^{-1}$, the freedom of description of NC Dp-branes translates in the matrix model like variables into how one separates the B-field into the external part B' and the internal part θ^{-1}. The internal part, θ^{-1}, is generated by the configuration of D(-1) instantons and B' corresponds to the external field imposed on them.

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